

Charged Topological Black Hole Pair Creation

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Abstract

I examine the pair creation of black holes in spacetimes with a cosmological constant of either sign. I consider cosmological C-metrics and show that the conical singularities in this metric vanish only for three distinct classes of black hole metric, two of which have compact event horizons on each spatial slice. One class is a generalization of the Reissner-Nordstrom (anti) de Sitter black holes in which the event horizons are the direct product of a null line with a 2-surface with topology of genus g . The other class consists of neutral black holes whose event horizons are the direct product of a null conoid with a circle. In the presence of a domain wall, black hole pairs of all possible types will be pair created for a wide range of mass and charge, including even negative mass black holes. I determine the relevant instantons and Euclidean actions for each case. Only for spherical are non-static solutions possible.

1 Introduction

Pair creation of black holes continues to afford us interesting insights into quantum gravity and the relationship between entropy and the number of quantum states of a black hole. It is a tunnelling process in which the mass-energy of the created pair of black holes is balanced by their negative potential energy in some background field, such as that of an electromagnetic field [1], a positive cosmological constant [2], a cosmic string [3] or a domain wall [4]. The amplitude for the process is approximated by e^{-I_i} , where I_i is the action of the relevant instanton *i.e.* a Euclidean solution to the field equations which interpolates between the states before and after a pair of black holes is produced.

The C metric solution of the Einstein-Maxwell equations [9] is interpreted as describing two oppositely-charged black holes undergoing uniform acceleration. It contains conical singularities, which in general cannot be eliminated at both poles. These singularities are interpreted as representing “rods” or “strings” which provide the force necessary to accelerate the black holes.

This raises the question as to what can provide the necessary force to accelerate the black holes. The simplest way is to remove the conical singularities by setting the acceleration to zero, in which case the C metric reduces to the Reissner-Nordström metric. A less trivial means of removing the conical singularities is to allow the black hole and acceleration horizons to coincide [1]; the Euclidean section of this metric has been referred to as the Type I instanton and has topology $S^2 \times R^2$. In this case the C metric has the same functional form as the Reissner-Nordström metric, with the coordinate that is usually regarded as the azimuthal coordinate playing the role of time. Analytically continuing $\phi \rightarrow i\Phi$ in the Euclidean Reissner-Nordström metric yields the type I instanton, which can be interpreted as representing the decay of a $(2 + 1)$ -dimensional Kaluza-Klein vacuum, $\text{Mink}^{(2,1)} \times S^1$ [2].

All other approaches have entailed adding additional forms of stress-energy to provide the necessary accelerating force. This typically generalizes the C metric to some other form, and it is the conical singularities of this modified metric that are then removed by some limiting procedure on its parameters. The first such approach taken [5] added a background electromagnetic field to the C metric using a Harrison transformation. This yielded the Ernst metric, which could be made regular by a suitable choice of electromagnetic field strength, providing the force necessary to accelerate the

black holes [1, 6]. It was subsequently shown that a cosmological constant can also do the job [2], the conical singularities being eliminated by fixing the acceleration parameter in terms of the cosmological constant. The resultant class of regular metrics are all of the Reissner-Nordström de Sitter type. Subsequent approaches have employed cosmic strings [3] and domain walls [4] and dilatonic generalizations [6, 7] to provide the necessary accelerating force.

In a recent paper [8] I demonstrated that the cosmological C -metrics [9] contain a rich array of Euclidean instantons that mediate pair production of black holes whose topology is of arbitrary genus. In this paper I consider a more detailed investigation of how these black holes can be derived from the C -metric and of their pair creation. The genus zero class of solutions has been studied before: it corresponds to the set of Reissner-Nordström de Sitter instantons that mediate cosmological black hole pair production [2]. All of the higher genus solutions are asymptotically anti-de Sitter, and correspond to instantons that are 4 dimensional generalizations of the 3 dimensional black hole [10]. Pair production of these black holes can take place in the presence of domain walls whose topology is the same as that of the produced black hole pairs. In a full theory of quantum gravity in which topology changing processes are expected to be important, these instantons will have to be taken into account.

The higher genus black hole spacetimes related to these instantons are of some interest in their own right. Like their topologically spherical cousins, they can form from gravitationally collapsing dust; however unlike their cousins, the initial density of the collapsing matter must be sufficiently small for collapse to take place [11]. A more recent investigation [12] of their thermodynamic behaviour has indicated that their entropy is one quarter their horizon area (although this result appears to be sensitive to the subtraction procedure used to make the Euclidean action finite [13]), and that they are thermodynamically stable. The ADM mass parameter of these black holes can even be zero [11, 14] or negative [15]; the negative mass hole can also form from the collapse of dust which violates the weak energy condition [15].

In section 2 I consider the cosmological form of the C -metric, and show that removal of its conical singularities yields the full array of topological black hole spacetimes noted above. I also find that two other qualitatively different black hole spacetimes can emerge, both of which can be considered generalizations of the spacetimes associated with the type-I instanton men-

tioned above. One of these has a non-compact event horizon. The other is a generalization of the $(3+1)$ dimensional constant curvature black hole recently considered by Banados [16]. In section 3 I discuss the basic properties of these topological black hole spacetimes, and compute their quasilocal mass and charge in the large-radius (ADM) limit. In section 4 I consider the pair creation of topological black holes using the domain wall mechanism [4, 17]. All topological black holes except those of spherical topology can only be pair-created in a static configuration. The pair-production rate for a black hole of arbitrary genus is computed in section 5. Constant curvature black hole pair creation is considered in section 6. A few concluding remarks are contained in the final section.

2 Cosmological C metrics

The cosmological C metric can be written as

$$ds^2 = \frac{1}{A^2(x-y)^2} \left[H(y)dt^2 - H^{-1}(y)dy^2 + G^{-1}(x)dx^2 + G(x)d\varphi^2 \right], \quad (1)$$

where

$$G(x) = \tilde{a} - bx^2 - 2mA x^3 - q^2 A^2 x^4 \quad (2)$$

and

$$H(y) = a - by^2 - 2mAy^3 - q^2 A^2 y^4 \quad (3)$$

with $\tilde{a} = a - \frac{\Lambda}{3A^2}$, Λ being the cosmological constant and A the acceleration parameter. The gauge field in the magnetic case is

$$F = -qdx \wedge d\varphi, \quad (4)$$

and the gauge field in the electric case is

$$F = -qdt \wedge dy, \quad (5)$$

where $q = \sqrt{r_+ r_-}$. The roots of $G(x)$ in ascending order will be denoted as x_1, x_2, x_3, x_4 ; similarly the roots of $H(y)$ in ascending order are y_1, y_2, y_3, y_4 .

A redefinition of the parameters a, b, m, q, A and Λ can be compensated for by a 3-parameter linear coordinate transformation which therefore maps

any C metric onto any other C metric up to a constant conformal transformation [18]. This freedom is typically used to eliminate the linear term in G and H and to set $a = 1$ and $b = 1$. Rather than using the entire 3-parameter freedom, I shall use only 1 parameter to eliminate the linear term, leaving a and b arbitrary, as has been assumed in (1). The parameter m will be assumed to be positive.

Under the coordinate transformation $y = x - 1/Ar$, $t = Au - \int^y dz/H(z)$, the metric (1) may also be written as

$$ds^2 = H(x - \frac{1}{Ar})A^2r^2du^2 - 2dudr - 2Ar^2dudx + r^2(G^{-1}(x)dx^2 + G(x)d\varphi^2) \quad (6)$$

The electric field becomes

$$F_E = -qdu \wedge (Adx + \frac{dr}{r^2}) \quad (7)$$

and the magnetic field is unchanged.

Before considering the elimination of conical singularities in (1), I shall first consider the structure of the polynomials G and H . In order that the metric (6) (or alternatively (1)) be of the appropriate signature, $G(x)$ must be positive over the allowed range of x . If there are no degenerate roots of $H(y)$, then either (a) $y < x$ or (b) $y > x$. When the parameters in the C metric are such that all roots of H are real, in case (a) the roots of H in ascending order (i.e. y_1, y_2, y_3) are respectively the inner black hole horizon, the outer black hole horizon, and the acceleration or cosmological horizon respectively, the largest root having no physical meaning. In case (b) the situation is reversed: the roots of H in descending order (i.e. y_4, y_3, y_2) are respectively interpreted as the inner black hole horizon, the outer black hole horizon, and the acceleration or cosmological horizon, the smallest root having no physical meaning.

The function G has at most three extrema. These are located at

$$x_m = \frac{-3m - \sqrt{9m^2 - 8q^2b}}{4q^2A}, x_0 = 0, x_p = \frac{-3m + \sqrt{9m^2 - 8q^2b}}{4q^2A}, \quad (8)$$

in ascending order if both $A > 0$, and $b < 0$ (i.e. $x_m < x_0 < x_p$). However if $A > 0$ and $b > 0$ then $x_m < x_p < x_0$. If $A < 0$ then both these inequalities are reversed.

Consider next the elimination of conical singularities in the (x, φ) sector. For definiteness, take $A > 0$. In order to have a regular solution, there must be no conical singularities at either of the endpoints of the domain of x , a criterion which can be satisfied in three ways.

(I) If the endpoints are a finite proper distance apart, then this criterion becomes[1]

$$G'(x_i) = -G'(x_{i+1}), \quad (9)$$

with φ periodically identified with period $\Delta\varphi = 4\pi/|G'(x_i)|$. Eq. (9) is equivalent to the condition

$$\prod_{j \neq i} |(x_i - x_j)| = \prod_{j \neq i+1} |(x_i - x_j)| \quad (10)$$

which can be satisfied by taking $x_i = x_{i+1}$ for $i = 1, 2, 3$. For $i = 1$ or $i = 3$, this is the only solution to (10). For $i = 2$, a non-trivial solution (with $x_i \neq x_{i+1}$) also exists, but would yield a metric which does not have the correct signature. Consequently the only solution to (9) is obtained by setting any adjacent pair of roots equal. This implies that $G(x)$ has an extremum at its (double) root. The conformal prefactor in front of the metric will therefore diverge only when $y = x_i = x_{i+1}$, which can be taken to be the upper or lower limit of the range of y as appropriate.

(II) There would be no conical singularity at $x = x_3$ if this point were an infinite proper distance from any allowed value of x . This condition requires that $x_2 = x_3$, implying that $G(x)$ has an extremum (a minimum) at its middle (double) root. In this case φ is still periodically identified with period $4\pi/|G'(x_4)|$, and the range of x is $x_2 = x_3 < x \leq x_4$. The (x, φ) sector is then no longer compact. The conformal prefactor will diverge unless the range of y is restricted to be less than $x_2 = x_3$. (Alternatively, one could choose $x_1 \leq x < x_2 = x_3$ in which case $y > x_2 = x_3$). Signature requirements then imply that $H(z) \leq G(z)$, *i.e.* the cosmological constant is negative or zero.

(III) Conical singularities could be eliminated if the middle pair of roots of $G(x)$ were complex. Although it would appear that the range of x is $x_1 \leq x \leq x_4$, in this case it is not possible to ensure finiteness of the conformal prefactor over the entire allowed range of y . However if $H(y)$ has a degenerate pair of roots at $y = y_D$, one could then perform a coordinate transformation on y such that the proper distance between these roots is finite. If such a degenerate pair exists, then $x_1 < y_D < x_4$ and the conformal prefactor in front of the metric will diverge at $x = y_D$. Avoiding this

divergence then entails restricting the range of x to $y_D < x \leq x_4$ rendering the (x, φ) sector noncompact as in the previous case. Conical singularities are removed by periodically identifying φ with period $4\pi/|G'(x_4)|$. (Alternatively one can choose $y_D > x \geq x_1$, in which case the period of φ must be $4\pi/|G'(x_1)|$ to avoid conical singularities). This case can only occur for negative cosmological constant.

Hence removal of conical singularities in the cosmological C metric implies that either $G(x)$ or $H(y)$ must have a double root. If such double roots are separated by a finite proper distance, the general analysis of such roots is carried out by considering the metric

$$d\ell^2 = \varepsilon \frac{dz^2}{F(z)} + F(z)d\Psi^2 \quad (11)$$

where $F(z) = -\prod_{i=1}^4 (z - z_i)$ is a quartic polynomial (either $G(x)$ or $H(y)$) and Ψ is either t ($\varepsilon = -1$) or φ ($\varepsilon = +1$). In the limit that a pair of adjacent roots (z_j, z_{j+1}) are equal, set

$$z = z_j + \epsilon f(\lambda) \quad z_{j+1} - z_j = \epsilon f(\hat{\lambda}) \equiv 2\epsilon\hat{\eta} \quad (12)$$

and take the limit as $\epsilon \rightarrow 0$. Rescaling $\Psi = \psi/\epsilon$ yields

$$d\ell^2 = -\varepsilon \frac{df^2}{\prod_{i \neq j, j+1}^4 (z_j - z_i) f(f - 2\hat{\eta})} - \prod_{i \neq j, j+1}^4 (z_j - z_i) f(f - 2\hat{\eta}) d\psi^2 \quad (13)$$

for the metric in this sector. Setting $f = \eta + \hat{\eta}/2$ yields

$$d\ell^2 = \varepsilon \frac{d\eta^2}{\prod_{i \neq j, j+1}^4 (z_j - z_i) (\hat{\eta}^2 - \eta^2)} + \prod_{i \neq j, j+1}^4 (z_j - z_i) (\hat{\eta}^2 - \eta^2) d\psi^2 \quad (14)$$

as the generic result. If either the largest or smallest pairs of roots of $F(z)$ are degenerate then $\prod_{i \neq j, j+1}^4 (z_j - z_i) > 0$, whereas if the middle two roots are degenerate then $\prod_{i \neq j, j+1}^4 (z_j - z_i) < 0$. Signature requirements will then dictate the range of $|\eta|$.

I consider next what kinds of spacetimes the metric (1) describes for each of the above cases.

2.1 Case I

Consider for definiteness, the situation when the double root is at $x = 0$. From (2) and (8) this implies that $\tilde{a} = 0$ or $A^2 = \frac{\Lambda}{3a}$. Note that solutions exist for both signs of Λ provided the signs of a and Λ match. The usual parameter choice $a = 1$ therefore eliminates all $\Lambda < 0$ solutions, and is therefore unnecessarily restrictive. Setting $\tilde{a} = k\epsilon^2$, the double-root condition is satisfied in the limit $\epsilon \rightarrow 0$. Writing $\varphi = \phi/\epsilon$ and $x = \epsilon f(\lambda)$, the roots $x_i = \epsilon f(\lambda_i)$ and $x_{i+1} = \epsilon f(\lambda_{i+1})$ coincide as $\epsilon \rightarrow 0$ for each of $i = 1, 2, 3$. From (14) the (x, φ) section of the metric becomes

$$\frac{dx^2}{G(x)} + G(x)d\varphi^2 = \frac{df^2}{k - bf^2} + (k - bf^2)d\phi^2 \quad (15)$$

in the limit $\epsilon \rightarrow 0$, apart from the conformal factor. Note that ϕ has period 2π .

The full metric is of the form

$$ds^2 = \frac{1}{A^2 y^2} \left[H(y) dt^2 - H^{-1}(y) dy^2 + d\Omega_b^2 \right], \quad (16)$$

which after making the further coordinate transformation $y = -\frac{1}{Ar}$, $t = AT$ becomes

$$ds^2 = -V(r)dT^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_b^2, \quad (17)$$

where

$$V(r) = -\frac{\Lambda}{3}r^2 + b - \frac{2m}{r} + \frac{q^2}{r^2}. \quad (18)$$

The range of r is from 0 to ∞ .

Degeneracy of the largest (or smallest) two roots of $G(x)$ occurs if and only if $b > 0$; in this case the class of metrics obtained are of the Reissner-Nordström (anti)-de Sitter type, of mass m and charge q . However there is a surprise in that the parameter b is completely arbitrary. A simple rescaling of parameters and coordinates allows it to be set to unit magnitude without loss of generality if it is nonzero (as is the case for k). If the middle two roots are degenerate, then $b = -1$, and if the largest (or smallest) three roots are degenerate then $b = 0$.

In all, there are five possible regular spacetimes which result, depending upon the relative signs of b , k and Λ . These are characterized by the two-dimensional metric $d\Omega_b^2$:

$$b = 1, k = 1 \quad d\Omega_b^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \quad (19)$$

$$b = 0, k = 1 \quad d\Omega_b^2 = d\theta^2 + \theta^2 d\phi^2 \quad (20)$$

$$b = -1, k = 1 \quad d\Omega_b^2 = d\theta^2 + \cosh^2(\theta)d\phi^2 \quad (21)$$

$$b = -1, k = -1 \quad d\Omega_b^2 = d\theta^2 + \sinh^2(\theta)d\phi^2 \quad (22)$$

$$b = -1, k = 0 \quad d\Omega_b^2 = d\theta^2 + e^{(2\theta)}d\phi^2 \quad (23)$$

where an obvious coordinate transformation $f = f(\theta)$ has been employed in each case. The $b = 1$ case is valid for both signs of Λ , and so corresponds to two different spacetimes, whereas signature requirements in the other four cases imply $\Lambda < 0$. The values of k given above are likewise determined by signature requirements.

It is easily checked that each of these six spacetimes satisfies the Einstein-Maxwell equations with cosmological constant. The gauge field becomes

$$F = -\frac{q}{r^2}dT \wedge dr \quad (24)$$

for all values of b in the electric case, and

$$b = 1, k = 1 \quad F = q \sin \theta d\theta \wedge d\phi \quad (25)$$

$$b = 0, k = 1 \quad F = q d\theta \wedge d\phi \quad (26)$$

$$b = -1, k = 1 \quad F = q \cosh \theta d\theta \wedge d\phi \quad (27)$$

$$b = -1, k = -1 \quad F = q \sinh \theta d\theta \wedge d\phi \quad (28)$$

$$b = -1, k = 0 \quad F = q e^\theta d\theta \wedge d\phi \quad (29)$$

in the magnetic case.

The preceding analysis holds in the case that the degenerate root is at $x = x_0 = 0$. It is straightforward to check that this analysis does not qualitatively change if the degenerate root is at either $x = x_m$ or $x = x_p$, and/or if the sign of A is reversed and/or if the roles of the largest and smallest roots are reversed, although the intermediate steps differ, as does the relationship between a , A and Λ . In other words, these six spacetimes are the only regular ones that can result from the requirement that the cosmological C-metric (1) be free of conical singularities for case I.

The regularity requirements for the C metric in the $b = 1$ case for positive Λ have been discussed previously in ref. [2]. The other five spacetimes, however, have been overlooked in previous studies. The $b = 1$, $\Lambda < 0$ case is simply Reissner-Nordström anti-de Sitter spacetime. The remaining three spacetimes all have $\Lambda < 0$ and $b < 0$, and necessarily have non-trivial topology, as I will demonstrate in the next section.

2.2 Case II

In this case the middle two roots of $G(x)$ are degenerate, and x ranges from its largest (smallest) root to this central degenerate root which is an infinite proper distance away. This implies that

$$G(x) = -q^2 A^2 (x - x_+) (x - x_0)^2 (x - x_-) \quad (30)$$

must hold, where by definition $x_+ > x_0 > x_-$. From (2), there can be no term linear in x in the right-hand-side of the above equation, which implies

$$x_0 = 0, \quad -2 \frac{x_- x_+}{x_+ + x_-} \quad (31)$$

and so there are two possible values for the central root. Note that for $\Lambda > 0$, $G(z) < H(z)$, so when the central two roots of $G(x)$ are degenerate $H(y)$ has only two roots y_- and y_+ . For $y_- < y < y_+$, y is a timelike coordinate, whereas for y outside these bounds the metric has a naked curvature singularity. Hence it is not possible to keep Λ positive and maintain the correct metric signature requirements without avoiding naked singularities.

If $x_0 = 0$ is the central root, then $0 < x_+$ and $x_- < 0$. For $A > 0$, a calculation then demonstrates that

$$x_{\pm} = \pm \frac{\sqrt{m^2 - q^2 b} \mp m}{q^2 A} \quad (32)$$

and the preceding inequalities hold provided $b < 0$ and $A^2 = \frac{\Lambda}{3a}$; the roles of x_{\pm} are interchanged if A reverses sign. After a coordinate transformation $x \rightarrow 1/(Ar)$ $y \rightarrow 1/(AR)$ $t \rightarrow AT$ and $\varphi \rightarrow \frac{2Ar_+^2}{r_+ - r_-} \phi$, the resultant metric is

$$\begin{aligned} ds^2 = & - \frac{(R^2/l^2 - 1 - 2\frac{m}{R} + \frac{q^2}{R^2}) r^2 dT^2}{(R+r)^2} + \frac{r^2 dR^2}{(R^2/l^2 - 1 - 2\frac{m}{R} + \frac{q^2}{R^2}) (R+r)^2} \\ & + \frac{r^2 R^2 dr^2}{(r^2 - 2mr - q^2) (R+r)^2} + \frac{4r_+^2 (r^2 - 2mr - q^2) R^2 d\phi^2}{(r_+ - r_-)^2 r^2 (R+r)^2} \end{aligned} \quad (33)$$

where $\Lambda = -3/l^2 < 0$ and $r_{\pm} = m \pm \sqrt{m^2 + q^2}$. The metric (33) is free of conical singularities in the (r, ϕ) section provided ϕ has period 2π . It satisfies the Einstein-Maxwell equations, where in the electric case

$$F = -\frac{q}{R^2} dT \wedge dR, \quad (34)$$

and

$$F = -\frac{q}{r^2} dr \wedge d\phi, \quad (35)$$

in the magnetic case.

If $x_0 = \frac{x_- x_+}{x_+ + x_-}$ is the central root, then $x_+ > 0$ and either $x_+ < -3x_-$ or $-\frac{x_+}{3} < x_- < 0$. The analysis of (30) is again straightforward (although somewhat more complicated) and yields (for $A > 0$)

$$x_{\pm} = \pm \frac{m(2\sqrt{1 + \sqrt{9 - \frac{8bq^2}{m^2}}} \mp (1 + \sqrt{9 - \frac{8bq^2}{m^2}}))}{4q^2 A} \quad (36)$$

and

$$x_0 = \frac{m(-3 + \sqrt{9 - \frac{8bq^2}{m^2}})}{4q^2 A} \quad (37)$$

provided

$$a = \frac{\Lambda}{3A^2} - \frac{27m^4 - 36bq^2m^2 + 8b^2q^4}{32q^6A^2} - \frac{m\sqrt{(9m^2 - 8q^2b)^3}}{32q^6A^2} \quad (38)$$

and $b > 0$. As a consequence $0 \leq q^2 \leq \frac{9m^2}{8}$. The metric is

$$ds^2 = -\frac{\hat{r}^2 \left(\frac{\hat{R}^2}{l^2} - \frac{(3\sqrt{g} - g)m^2}{4q^2} + \frac{(1 - \sqrt{g})m}{\hat{R}} + \frac{q^2}{\hat{R}^2} \right) d\hat{T}^2}{(\hat{R} + \hat{r})^2} + \frac{\hat{r}^2 d\hat{R}^2}{(\hat{R} + \hat{r})^2 \left(\frac{\hat{R}^2}{l^2} - \frac{(3\sqrt{g} - g)m^2}{4q^2} + \frac{(1 - \sqrt{g})m}{\hat{R}} + \frac{q^2}{\hat{R}^2} \right)} \quad (39)$$

$$+ \frac{\hat{r}^2 \hat{R}^2 d\hat{r}^2}{\left(\frac{(3\sqrt{g} - g)m^2 \hat{r}^2}{4q^2} + (1 - \sqrt{g})m\hat{r} - q^2 \right) (\hat{R} + \hat{r})^2} \quad (40)$$

$$+ \frac{\hat{R}^2 ((3\sqrt{g} - g) \frac{m^2}{4q^2} \hat{r}^2 + (1 - \sqrt{g})m \hat{r} - q^2) d\hat{\phi}^2}{r^2 (\hat{R} + \hat{r})^2} \quad (41)$$

where $g \equiv 9 - \frac{8bq^2}{m^2}$. Since $0 < g < 9$, the term $\lambda^2 \equiv \frac{(3\sqrt{g}-g)m^2}{4q^2} > 0$. Rescaling the coordinates $(\hat{T}, \hat{R}, \hat{r}, \hat{\phi}) \rightarrow (T/\lambda, \lambda R, \lambda r, \tilde{\phi}/\lambda)$ then yields

$$\begin{aligned} ds^2 = & - \frac{r^2 \left(\frac{R^2}{l^2} - 1 + \frac{(1 - \sqrt{g})m}{\lambda^3 R} + \frac{q^2}{\lambda^4 R^2} \right) dT^2}{(R + r)^2} \\ & + \frac{r^2 dR^2}{(R + r)^2 \left(\frac{R^2}{l^2} - 1 + \frac{(1 - \sqrt{g})m}{\lambda^3 R} + \frac{q^2}{\lambda^4 R^2} \right)} \\ & + \frac{r^2 R^2 dr^2}{\left(r^2 + \frac{(1 - \sqrt{g})m}{\lambda^3} r - \frac{q^2}{\lambda^4} \right) (R + r)^2} + \frac{R^2 (r^2 + \frac{(1 - \sqrt{g})m}{\lambda^3} r - \frac{q^2}{\lambda^4}) d\tilde{\phi}^2}{r^2 (R + r)^2} \quad (42) \end{aligned}$$

which is qualitatively the same as the metric (33) once m and q are rescaled. However the coefficient $(1 - \sqrt{g})$ multiplying m can be either positive or negative.

Hence the general form of the case II metric is

$$\begin{aligned} ds^2 = & - \frac{(R^2/l^2 - 1 - 2M/R + Q^2/R^2) r^2 dT^2}{(R + r)^2} + \frac{r^2 dR^2}{(R^2/l^2 - 1 - 2M/R + Q^2/R^2) (R + r)^2} \\ & + \frac{r^2 R^2 dr^2}{(r^2 - 2Mr - Q^2) (R + r)^2} + \frac{4r_+^2 (r^2 - 2Mr - Q^2) R^2 d\phi^2}{(r_+ - r_-)^2 r^2 (R + r)^2} \quad (43) \end{aligned}$$

where m and q have been appropriately rescaled, and ϕ has been rescaled so as to removed spurious conical singularities. The parameter M may have either sign. The electromagnetic field strength tensors are the same as (34) and (35) respectively with $q \rightarrow Q$.

The coordinate R has the range $0 \leq R \leq \infty$ whereas signature requirements demand that the coordinate r have the range $r_+ \leq r \leq \infty$, where $r_{\pm} = M \pm \sqrt{M^2 + Q^2}$. reminiscent of the type I instanton discussed in ref. [2]. For large r the metric (33) asymptotically approaches the metric (17) where $d\Omega_b^2$ is given by (23). Near $r = r_+$ the metric is conformal to (17), where $d\Omega_b^2$ is given by (20). There is a curvature singularity at $R = 0$, and event horizons at the roots of the equation $R^2/l^2 - 1 - 2M/R + Q^2/R^2 = 0$. These horizons are not compact surfaces.

2.3 Case III

This case is similar to case I, but with the roles of (x, φ) and (y, t) reversed.

Again, suppose for definiteness that $H(y)$ has a double root at $y = 0$. From (3) and (8) this implies that $a = 0$. Setting $a = k\epsilon^2$, and writing $T = t/\epsilon$ and $y = \epsilon R(\lambda)$, the roots $y_i = \epsilon R(\lambda_i)$ and $y_{i+1} = \epsilon R(\lambda_{i+1})$ coincide as $\epsilon \rightarrow 0$ for each of $i = 1, 2, 3$. From (14) the (t, y) section of the metric becomes

$$-\frac{dy^2}{H(y)} + H(y)dy^2 = -(bR^2 - k)dT^2 + \frac{dR^2}{bR^2 - k} \quad (44)$$

as $\epsilon \rightarrow 0$, apart from the conformal factor. Signature requirements permit all possible signs for b and k , although for $b < 0$ and $k > 0$ R is a timelike coordinate and the metric is no longer static.

After making the coordinate transformation $x = -\frac{1}{Ar}$, $\varphi = A\hat{\phi}$ the full metric becomes

$$ds^2 = r^2 \left(-(bR^2 - k)dT^2 + \frac{dR^2}{bR^2 - k} \right) + \frac{dr^2}{U(r)} + U(r)d\hat{\phi}^2 \quad (45)$$

where

$$U(r) = \frac{|\Lambda|}{3}r^2 - b + \frac{2m}{r} - \frac{q^2}{r^2}. \quad (46)$$

Provided that either $q^2 > \frac{b^2 l^2}{12}$ or $m > m_+ \equiv \frac{\hat{r}_+}{2}(\frac{\hat{r}_+^2}{l^2} + \frac{q^2}{\hat{r}_+^2})$ or $m < m_- \equiv \frac{\hat{r}_-}{2}(\frac{\hat{r}_-^2}{l^2} + \frac{q^2}{\hat{r}_-^2})$ where

$$\hat{r}_\pm^2 = \frac{1}{6}l^2 b(1 \pm \sqrt{1 - 12 \frac{q^2}{b^2 l^2}}) \quad (47)$$

the function $U(r)$ will have only one positive root $r = r_U$, and the range of r is $r_U < r < \infty$. If $m_- < m < m_+$ then $U(r)$ will have three positive roots, and the range of r will be $r_M < r < \infty$ where r_U is now the largest root of $U(r)$. In both cases conical singularities are not present provided $\hat{\phi}$ has period $\frac{4\pi}{|U'(r_U)|}$. Although signature requirements also permit r to lie between the smallest two positive roots of $U(r)$, it is not possible to eliminate conical singularities in the $(r, \hat{\phi})$ section.

The magnetic gauge field (4) becomes

$$F = -\frac{q}{r^2}dr \wedge d\hat{\phi}, \quad (48)$$

which is like an electric field, whereas the electric gauge field (5) is now

$$F = -q dR \wedge dT, \quad (49)$$

is like a magnetic field where, for example, $R = \cos(\chi)$ when $k = b = -1$, yielding $F = q \sin(\chi) d\chi \wedge dT$.

The class of metrics (45) are products of a 2 dimensional (anti) de Sitter spacetime (which is a black hole for $b > 0$ and $k > 0$ [19]) with a Euclidean 2 dimensional de Sitter space. Employing the coordinate transformation $\rho^2 = U(r)l^2$, these metrics become

$$ds^2 = [U^{(-1)}(\rho^2/l^2)] \left(-(bR^2 - k)dT^2 + \frac{dR^2}{bR^2 - k} \right) + \frac{d\rho^2}{U'(U^{(-1)}(\rho^2/l^2))} + \rho^2 d\phi^2 \quad (50)$$

where $\hat{\phi} = 2l|U'(r_U)|\phi$ and $U^{(-1)}$ is the inverse of U , i.e. $U(U^{(-1)}(z)) = z$. For large r , $\rho \approx r$, and the metric and these metrics are all asymptotic to

$$ds^2 = (\rho^2 + bl^2) \left(-(bR^2 - k)dT^2 + \frac{dR^2}{bR^2 - k} \right) + \frac{d\rho^2}{\frac{\rho^2}{l^2} + b} + \rho^2 d\phi^2 \quad (51)$$

which is the product of $(2+1)$ (anti) de Sitter spacetime and a circle. When $m = q = 0$ the metric (51) is exactly equal to (45) after the coordinate transformation $\rho^2 = r^2 - bl^2$. For $b < 0$ these are $(3+1)$ dimensional versions of the constant curvature black holes recently considered by Banados [16]. If either of m or q are non-zero, the metric (50) has naked singularities.

If the degenerate roots of $H(y)$ are not at $y = 0$, the situation is similar to the situation just described, and the resultant class of metrics is still given by (45), but with Λ , m and b redefined. However the analysis is somewhat more complicated and will not be reproduced here.

3 Topological Anti de Sitter Black Holes

The case I metrics (17) yield an interesting class of topological black holes which I shall describe in this section.

The $b = 1$ metrics (19) correspond to the usual Reissner-Nordstrom de Sitter and Reissner-Nordstrom anti de Sitter spacetimes respectively, depending upon the sign of Λ . The event horizons have the topology of a 2-sphere.

The remaining spacetimes all have $\Lambda < 0$ and $b \leq 0$. Surfaces of constant T and r (including the event horizon) are apparently noncompact spaces of constant nonpositive curvature. For $b = 0$ the existence of such “black plane” spacetimes have recently been noted [20]. However by appropriate identification of the coordinates it is possible to render these surfaces compact for all $b \leq 0$. Since they are also surfaces of constant negative curvature they must have a non-trivial topology, which is in turn inherited by the entire spacetime.

Consider first $b = 0$. The (θ, ϕ) section is flat space. Geodesics in this space are straight lines. By a trivial coordinate transformation the metric in this section may be written as

$$d\Omega_b^2 = d\lambda^2 + d\tilde{\phi}^2 \quad (52)$$

where the geodesics are given by the equations $\alpha_1\lambda + \alpha_2\phi = \alpha_3$ where the α_i are constants. Identifying λ and $\tilde{\phi}$ with their own periodicities the (θ, ϕ) sector becomes a torus. Its unit area can be chosen to be 4π by identifying the λ -coordinate with period 2 and the $\tilde{\phi}$ coordinate with period 2π .

The remaining spacetimes (21), (22), (23) all have $b = -1$. The (θ, ϕ) sections are non-compact and respectively have the topologies of a tube flared out at both ends, a two-sheeted hyperboloid, and a tube flared at one end.

Consider first the spacetime (22), taking one sheet of the hyperboloid. The (θ, ϕ) section may be mapped to the Poincaré disk under the transformation $\lambda = \tanh(\theta/2)$, yielding

$$d\theta^2 + \sinh^2(\theta)d\phi^2 = \frac{1}{(1-\lambda^2)^2} (d\lambda^2 + \lambda^2 d\phi^2) \quad (53)$$

where $0 \leq \lambda < 1$. The Poincaré disk has an isomorphism group $SO(2,1)$. Identifying points on the disk under any discrete subgroup of $SO(2,1)$ yields a compact two-dimensional space of negative curvature, which necessarily has genus $g \geq 2$. The unit area of such surfaces is $4\pi(g-1)$. These spaces may be constructed by symmetrically placing a polygon of $4g$ sides at the center of the Poincaré disk and identifying opposite sides. The edges of the polygon are geodesics of the Poincaré disk; these are circles whose extensions are orthogonal to the disk boundary. The simplest case is the octagon with $g = 2$. The $q = m = 0$ versions of these spacetimes can be understood as four-dimensional analogues of the three-dimensional black hole [10], as shown recently by Aminneborg *et.al.* [14].

Carrying out an analogous procedure on the other two spacetimes (23) and (22) yields nothing new. This is because each of their (θ, ϕ) sections can locally be mapped into the (θ, ϕ) section of (22). For example the local transformation

$$\sinh(\sigma) = \cos(\phi) \sinh(\theta) \quad \tanh(\Phi) = \sin(\phi) \tanh(\theta) \quad (54)$$

yields

$$d\sigma^2 + \cosh^2(\sigma) d\Phi^2 \rightarrow d\theta^2 + \sinh^2(\theta) d\phi^2 \quad (55)$$

whereas

$$\begin{aligned} \chi &= \frac{\sinh(\theta) \sin(\phi)}{\cosh(\theta) + \sinh(\theta) \cos(\phi)} \\ \alpha &= \ln(\cosh(\theta) + \sinh(\theta) \cos(\phi)) \end{aligned} \quad (56)$$

yields

$$d\alpha^2 + \exp(2\alpha) d\chi^2 \rightarrow d\theta^2 + \sinh^2(\theta) d\phi^2 \quad . \quad (57)$$

Hence any local region in the (θ, ϕ) section of (22) containing the $4g$ -sided polygon can, along with the polygon, be mapped into a local region of either of (21) or (23). The identification procedure then follows through. Similarly, The electromagnetic field tensors (27) and (29) correspondingly map into (28). Hence without loss of generality the spacetimes (21) and (23) may be dropped from further consideration. These $b \leq 0$ constructions hold for all values of r and T in (17).

For $\Lambda < 0$, the metric function $V(r)$ in (18) has no term linear in r , and the product of its roots is equal to $q^2 l^2$. Hence $V(r)$ has at most two roots for positive r corresponding to an inner and outer horizon, as with the usual RNadS metric. For $b = 0$, provided

$$27 l^2 m^4 \geq 16 q^6 \quad (58)$$

there will be two horizons, with the extremal case saturating the inequality. For nonzero b event horizons exist provided

$$m^2 \leq \frac{l^2}{27} \frac{16 - 24 e^2 b - 16 b \sqrt{1 - e^2 b} e^2 + 6 b^2 e^4 + 16 \sqrt{1 - e^2 b}}{e^6} \quad (59)$$

where $e = \frac{2\sqrt{2}q}{3m}$. If $b = 1$ (the genus 0 case) a necessary condition for event horizons to exist is that $q < m$, since for $q > m$ the right-hand side of (59)

becomes negative and so (59) cannot be satisfied; the range of e is therefore from 0 to $2\sqrt{2}/3$. If $b = -1$ then there is no (obvious) upper limit on e , and event horizons can exist for arbitrarily large values of q relative to m .

Hence for all values of b , the topology of the outer event horizon is H_g^2 , where H_g^2 is a two-dimensional surface of genus g , with $g = 0$ being the 2-sphere. The entire spacetime has topology $R^2 \times H_g^2$.

An analysis of the evaluation of the mass and charge of the metrics given by (17) may be carried out using the quasilocal formalism developed for anti de Sitter spacetimes [21]. Consider a surface B_g of topology H_g at a fixed value R of the coordinate r centered about the origin. Since $\partial/\partial t$ is a surface-forming Killing vector proportional to the normal of B_g , the conserved mass parameter is given by

$$M = \int_{B_g} d\Omega_g \sqrt{V(R)} \left(\frac{k}{\kappa} - \epsilon_0 \right) \quad (60)$$

where $\kappa = 8\pi$ is the gravitational coupling constant and k is the trace of the extrinsic curvature of the surface B_g considered as a boundary $\partial\Sigma_g$ of a spacelike hypersurface Σ_g whose unit normal is orthogonal to the normal of B_g . The quantity ϵ_0 is the energy density associated with some reference spacetime. Although it is not unique, in order to make a meaningful comparison for a given topology, the reference spacetime should be chosen to be a spacetime with the same topology as the original spacetime. The natural choice would be a spacetime with $m = 0 = q$ – these are the massless AdS black holes considered in refs. [11, 14]. The trace of the extrinsic curvature for the boundary for $\Lambda = -3/l^2$ is

$$k = -\frac{2}{R} \sqrt{R^2/l^2 + b - 2m/R + q^2/R^2} \quad (61)$$

independent of the topology. Taking ϵ_0 to be equal to k when $m = 0 = q$ yields in the limit of large R

$$M = \frac{m}{4\pi} \int_{B_g} d\Omega_g = m(|g-1| + \delta_{g,1}) \quad (62)$$

as the conserved ADM mass of the spacetime. A similar analysis of the charge Q contained within the same boundary B_g indicates that $Q = q(|g-1| + \delta_{g,1})$ is the conserved charge of the black hole.

It is straightforward to show [21] that the quantity M in (60) is simply the Hamiltonian derived from the action

$$S = -\frac{1}{2\kappa} \int d^4x \sqrt{g} (R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \frac{1}{\kappa} \int_{\Sigma_i}^{\Sigma_f} d^3x \sqrt{h} K, -\frac{1}{\kappa} \int_{\mathcal{T}} d^3x \sqrt{\gamma} \Theta - S_0 \quad (63)$$

evaluated when the constraints hold, in the large R limit. Here $\kappa = 8\pi$, and $\mathcal{T} = B \times I$ is a timelike hypersurface (with induced metric γ_{ij} and extrinsic curvature Θ_{ij}) joining the initial and final hypersurfaces Σ_i and Σ_f (with induced metric h_{ij} extrinsic curvature K_{ij}) respectively. S_0 is the reference action which yields ϵ_0 , and is a functional of the metric on the boundary.

4 Topological Black Hole Instantons

From the preceding sections it is clear that the only special cases of the cosmological C metrics for which the metric is regular and the event horizon is compact reduce either to one of the Reissner-Nordström (anti) de Sitter metrics (17) or the constant curvature black holes (51). A consideration of the pair creation of black holes then reduces to a consideration of the non-singular instantons that can be constructed from either of these cases. The remaining metrics have either naked singularities (50) or non-compact event horizons (43). The possibility of compactifying these horizons will not be considered here.

For the Reissner-Nordstrom (anti) de Sitter (RN(a)dS) spacetimes the general form of the metric can be written as

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega_b^2, \quad (64)$$

where

$$N(r) = -s \frac{r^2}{l^2} + b - \frac{2m}{r} + \frac{q^2}{r^2} \quad (65)$$

with $l^2 = \frac{3}{|\Lambda|}$, $s = \frac{|\Lambda|}{\Lambda}$ is the sign of Λ , and

$$d\Omega_b = \left\{ \begin{array}{ll} d\theta^2 + \sin^2(\theta)d\phi^2 & b = 1, s = \pm 1 \\ d\theta^2 + d\phi^2 & b = 0, s = -1 \\ d\theta^2 + \sinh^2(\theta)d\phi^2 & b = -1, s = -1 \end{array} \right\} \quad (66)$$

corresponding to the genus $g = 0$, $g = 1$ and $g \geq 2$ cases respectively. The range of r is from 0 to ∞ . The gauge field is

$$F = -\frac{q}{r^2} dt \wedge dr \quad (67)$$

for an electrically-charged solution, and

$$F = \left\{ \begin{array}{ll} q \sin(\theta) d\theta \wedge d\phi & b = 1, s = \pm 1 \\ q d\theta \wedge d\phi & b = 1, s = -1 \\ q \sinh(\theta) d\theta \wedge d\phi & b = 1, s = -1 \end{array} \right\} \quad (68)$$

in the magnetic case.

I shall restrict my attention to magnetically charged black holes, and later consider electrically charged ones. Instantons can be constructed from the metric (64) by analytically continuing $t \rightarrow i\tau$. In order to obtain a positive-definite metric the coordinate r must lie in a region that ensures $N(r) > 0$. For values of the coordinate $r \geq r_H$ (where r_H is the outer horizon) such that $N(r_H) = 0$ there is potentially a conical singularity at this point. If $N'(r_H) \neq 0$, then a necessary condition for the regularity of the instanton is that this conical singularity be removed by making τ periodic with period

$$\beta_H = \frac{2\pi}{\kappa_H} = \frac{4\pi r_H}{-\frac{3sr_H^2}{l^2} + b - \frac{q^2}{r_H^2}} \quad (69)$$

where κ_H is the surface gravity at $r = r_H$. If $\kappa_H = 0$ then this point is an infinite proper distance from any other point, and so $r > r_H$ and τ can be identified with arbitrary period [7].

Since $N(r)$ diverges for large r , these instantons will only be regular provided one of the following additional conditions is satisfied.

- (A) If $N \rightarrow -\infty$ then $N(r)$ must have another root $r = r_C > r_H$, interpreted as a cosmological horizon. In this case a regular instanton can be obtained by either (i) identifying τ with period $2\pi/\kappa_C$ (when $\kappa_H = 0$) or (ii) setting $|\kappa_C| = |\kappa_H|$ (when $N'(r_H) \neq 0$) so that the periodicity at both horizons is the same.
- (B) If $N \rightarrow \infty$ then there are no other roots of $N(r)$ for the metric (64). The instanton, although regular, is not compact, and must be modified by including additional stress energy in order to ameliorate this difficulty.

Case (A) is the de Sitter case ($s = 1$), and a discussion of its instantons has already appeared [2]. Here I briefly recapitulate the results. Since there are two horizons, there are in all four types of instantons:

- (a) lukewarm, when $|\kappa_C| = |\kappa_H|$ but $r_C \neq r_H$ (implying $q = m$)
- (b) charged Nariai, when $|\kappa_C| = |\kappa_H|$ and $r_C = r_H$
- (c) cold, if $\kappa_H = 0$ and $r_C \neq r_H$
- (d) ultracold, if $\kappa_H = 0$ and $r_C = r_H$

(There is a second ultracold instanton, but it does not have horizons [2]). For sufficiently small mass, both lukewarm and cold instantons can exist, which respectively correspond to pair creation of non-extreme and extreme black holes. At given mass, the cold solution has higher charge than the lukewarm solution. For $m \geq l/(3\sqrt{3})$, the charged Nariai instanton is viable; it has lower charge than the other two. For $m \geq 3l/4$, the lukewarm and charged Nariai solutions coincide, and there is no lukewarm solution with higher mass. The cold and charged Nariai solutions coincide in the ultracold solution, when $m = 2l/(3\sqrt{6})$, and there are no regular solutions where the mass is larger than this.

Case (B) is the anti de Sitter case. As previously noted [8] pair production of these black holes may be achieved using domain walls [4, 17]. Since the gravitational field of a domain wall is repulsive, inclusion of a domain wall into the anti de Sitter case can provide the necessary energy to pair create the black holes, analogous to the manner in which the positive cosmological constant performs a similar function in the de Sitter case. Of course there is nothing obstructing the inclusion of domain walls in the de Sitter case, and it can be included in the analysis as well.

The general situation involves constructing a two-sided bubble by taking two regions of the RNadS spacetime and joining them together along a common timelike boundary which is homeomorphic to $H_g^2 \times R$. The boundary along which they are joined must satisfy the Israel matching conditions. In the Lorentzian section the result is a spacetime with two domains, each of which contains a black hole. The topology of the RNadS Riemannian section is $R^2 \times H_g^2$, where the R^2 factor is like a bell. Two copies of this manifold may be matched together at a radius r at the open end of their bells

determined by the matching condition. The resultant Riemannian section now has topology $S^2 \times H_g^2$ which contains a single domain wall of topology $S^1 \times H_g^2$ and two bolts of topology H_g^2 where the Killing field $\frac{\partial}{\partial \tau}$ vanishes. The nucleation surface Σ which joins the Lorentzian and Riemannian sections is located along the $\tau = 0$ and $\tau = \beta_H/2$ segments.

The matching condition for the class of spacetimes given by (64) may be obtained in a manner completely analogous to the genus zero cases [22, 23] and is given by

$$\sqrt{N(r) - \dot{r}^2} = 2\pi\sigma r \quad (70)$$

where σ is the energy per unit area of the domain wall, whose topology is $S^1 \times H_g^2$, and the overdot refers to the derivative with respect to Euclidean proper time. Equation (70) may be interpreted as the equation describing the motion of a fictitious particle in a potential $v = N - (2\pi\sigma r)^2$.

Static solutions for which $r = r_s$ have energy zero, and may be obtained by solving (70) under the condition $\partial v / \partial r = 0$. These are given by

$$r_s^2 = \frac{l^2}{6\gamma} \left[b \pm \sqrt{b^2 - 12 \frac{q^2 \gamma}{l^2}} \right] \quad (71)$$

where $\gamma \equiv (2\pi\sigma l)^2 + s$. There are several different possible solutions depending upon the signs and magnitudes of γ and b .

$$\begin{aligned} b > 0 \quad \gamma > 0 \quad r_s^2 &= \frac{l^2}{6\gamma} \left[1 + \sqrt{1 - 12 \frac{q^2 \gamma}{l^2}} \right] \\ m^2 &= \frac{2}{3} q^2 + \frac{l^2}{54\gamma} \left[1 + \left(1 - 12 \frac{q^2 \gamma}{l^2} \right)^{3/2} \right] \end{aligned} \quad (72)$$

$$\begin{aligned} b > 0 \quad \gamma = 0 \quad r_s^2 &= q^2 \\ m &= q \end{aligned} \quad (73)$$

$$\begin{aligned} b > 0 \quad \gamma < 0 \quad r_s^2 &= \frac{l^2}{6|\gamma|} \left[-1 + \sqrt{1 + 12 \frac{q^2 |\gamma|}{l^2}} \right] \\ m^2 &= \frac{2}{3} q^2 + \frac{l^2}{54|\gamma|} \left[-1 + \left(1 + 12 \frac{q^2 |\gamma|}{l^2} \right)^{3/2} \right] \end{aligned} \quad (74)$$

$$b = 0 \quad \gamma < 0 \quad r_s^2 = \frac{|q|l}{\sqrt{3|\gamma|}} = \frac{2q^2}{3m}$$

$$m^2 = \frac{4q^3\sqrt{|\gamma|}}{3\sqrt{3}l} \quad (75)$$

$$b < 0 \quad \gamma < 0 \quad r_s^2 = \frac{l^2}{6|\gamma|} \left[1 + \sqrt{1 + 12\frac{q^2|\gamma|}{l^2}} \right]$$

$$m^2 = -\frac{2}{3}q^2 + \frac{l^2}{54|\gamma|} \left[1 + \left(1 + 12\frac{q^2|\gamma|}{l^2} \right)^{3/2} \right] \quad (76)$$

For all solutions with $b \neq 0$, $r_s = \frac{3m}{2} \left(b + \sqrt{1 - b\frac{8q^2}{9m^2}} \right)$.

These are the only allowed solutions. Since every solution of the matching condition must obey $3m = r_s(b + \frac{q^2}{r_s^2})$, for $b \leq 0$ and $m > 0$ there is no solution with zero charge, and so it will not be possible to pair-create neutral black holes with non-trivial topology. However for $b < 0$ and $m < 0$ static solutions of zero charge exist with $r_s = 3|m|$. This will correspond to the pair creation of neutral negative mass black holes [15]. Static charged solutions of negative mass also exist, with $r_s = 3|m|\frac{1+\sqrt{1+e^2}}{2}$.

Non-static solutions of (70) describe the creation of accelerating black holes and are periodic in the Euclidean time τ . This period is

$$\beta_W = \oint_{r_{\min}}^{r_{\max}} d\tau = \oint_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{V(V - (2\pi\sigma r)^2)}} \quad (77)$$

and is the amount of Euclidean time needed for the wall to interpolate between the turning points r_{\min} and r_{\max} of the motion (70), both of which must be real and positive. The wall will intersect itself unless it moves between the turning points an integral number of times within the period β_H . Hence

$$\beta_W = \frac{\beta_H}{n} \quad (78)$$

where n is a positive integer.

An analysis of the turning points of the motion involves a consideration of the function

$$N - (2\pi\sigma r)^2 \equiv U(r) = -\frac{\gamma}{l^2}r^2 + b - 2\frac{m}{r} + \frac{q^2}{r^2} \quad (79)$$

for regions $r_{\min} \leq r \leq r_{\max}$ such that $U(r) \geq 0$. This means that $U(r)$ must have at least three roots for positive r . As with the metric function

$V(r)$ in (18), there are only two roots of $U(r)$ if $\gamma < 0$, whereas if $\gamma \geq 0$ there are three roots only if $b > 0$. An alternative way of seeing this is by realizing that $U(r) \geq 0$ implies

$$q^2 - 2mr \geq \frac{\gamma}{l^2} r^4 - br^2 \quad . \quad (80)$$

The left hand side of this equation is a straight line of slope $-2m$ and intercept q^2 , whereas the right hand side is a quartic intersecting the origin and symmetric about $r = 0$. For negative γ , the quartic can intersect the line for positive r at two points at most; however in between these regions the inequality is violated unless the points coincide, in which case inequality is saturated. For positive γ this is also true unless b is positive, in which case there exists a local maximum of the quartic at the origin, yielding a region $r > 0$ in which the inequality is satisfied (or alternatively saturated at two distinct points), provided γ/l^2 is sufficiently small. This translates into a bound on the mass which is $\frac{1}{3\sqrt{3}} \leq \frac{ml}{\sqrt{\gamma}} \leq \frac{\sqrt{2}}{3\sqrt{3}}$. However from (59), there are no horizons for $q > m$. This shifts the upper bound on m downward, so that $\frac{1}{3\sqrt{3}} \leq \frac{ml}{\sqrt{\gamma}} \leq \frac{1}{4}$.

Hence the only accelerating black holes that can be pair-created by domain walls are those of spherical topology, in either de Sitter or anti de Sitter space, and the latter scenario is possible only if σ is sufficiently large. Only static topological black holes can be pair created using domain walls. This conclusion holds regardless of the sign of m . Indeed, in order to pair-create negative mass holes, both b and γ must be negative.

The preceding conclusions are altered if the sign of q^2 is reversed, as is sometimes done in considering electrically charged black holes instead of magnetic ones [24]. Analytically continuing $q \rightarrow iq$ will reverse the scenario described above. The only accelerating black holes which could be pair-created would be those of negative mass and genus $g \geq 2$ topology, and would have $b < 0$ and $\gamma < 0$. This would violate electromagnetic duality. However instead of continuing the charge to imaginary values, it is possible to consider electrically charged instantons in which the electromagnetic field is pure imaginary on the Riemannian section. This restores duality [2, 25], and leads to the same pair creation scenarios and production rates that the magnetically charged holes have.

5 Topological Black Hole Pair Production

Since the condition (77) is only applicable to black holes of spherical topology, a more detailed analysis of its validity is identical to that considered for supergravity domain walls [17] with equal negative values of the cosmological constant on either side of the wall. There is a countably infinite set of instantons which satisfy (77) which can mediate the creation of accelerating spherical black hole pairs from the initial domain wall state.

All other black hole pairs will be created in a static configuration satisfying one of the conditions (72) – (76). I shall consider only the pair creation of magnetically charged static black holes; the electric case yields the same results, but entails the incorporation of an additional surface term that vanishes in the magnetic case [2, 25].

The Euclidean action for these instantons is

$$I = \int d^4x \sqrt{g} \left(-\frac{R}{16\pi} + \frac{F^2}{16\pi} + \mathcal{L}_c + \mathcal{L}_d \right) \quad (81)$$

where \mathcal{L}_c is the cosmological Lagrangian and \mathcal{L}_d the domain-wall Lagrangian. The former may be taken to be that of the squared field strength of a 3-form or simply the constant $3\frac{s}{8\pi l^2}$. The domain wall Lagrangian can be that of a membrane current coupling to the 3-form [23] or that of a scalar field Φ whose potential $\mathcal{V}(\Phi)$ is everywhere positive [4] (and so its Euclidean action is always negative). There are no boundary terms because the instantons considered here are compact and without boundary.

Regardless of the mechanism, the Einstein field equations applied to (81) yield

$$\begin{aligned} I &= \int_{M_e} d^4x \sqrt{g} \left(-\frac{R}{16\pi} + \frac{3s}{8\pi l^2} + \frac{F^2}{16\pi} \right) - \frac{\sigma}{2} \int_W \sqrt{h} d^3x \\ &= \int d^4x \sqrt{g} \left(-\frac{3s}{8\pi l^2} + \frac{F^2}{16\pi} \right) - \frac{\sigma}{2} \int_W \sqrt{h} d^3x \end{aligned} \quad (82)$$

for the Euclidean action, where σ is the energy density of the wall, and the Euclidean section M_e includes the volumes on both sides of the wall. For a pair of genus g black holes of mass m and charge q

$$I(m, q, g) = \beta_H(|g-1| + \delta_{g,1}) \left(-\frac{s(r_s^3 - r_H^3)}{l^2} + \frac{q^2}{r_s r_H} (r_s - r_H) - 2\pi\sigma r_s^2 \sqrt{N(r_s)} \right) \quad (83)$$

where r_H is the location of the outer horizon of the black hole, β_H is given by (69), and r_s is given by the relevant equation in (72) – (76).

The amplitude for pair creation will be approximately $e^{-I/2}$ because half the Euclidean section provides an instanton for the pair creation of black holes. Hence the rate of pair creation will be proportional to the probability e^{-I} . What is physically meaningful is a comparison of the creation rate of the combined black hole-wall configuration relative to the creation rate an appropriate background configuration. For genus 0 black holes this background can be taken to be that of a domain wall with empty adS space on either side, obtained by gluing two hyperbolic 4-balls along their boundary 3-spheres. For higher genus black holes this is somewhat problematic, as there are several choices of a comparative background, depending upon how one chooses to view the topological black hole creation process.

Within a given topological sector of genus g , the domain wall must have the same topology in order to satisfy the matching conditions. However a domain wall of this topology cannot be matched to an empty adS space. The natural background within a given topological sector would seem to be the $m = q = 0$ configuration. In this case the relative pair-creation rate would be

$$\begin{aligned}\Gamma_g &= e^{-I(m,q,g)+I(0,0,g)} \\ &= \exp \left[|g-1| \left(\frac{2(2\pi\sigma l)^2 - 1}{[(2\pi\sigma l)^2 - 1]^{3/2}} - \beta_H \left(\frac{(r_s^3 - r_H^3)}{2l^2} + \frac{q^2}{r_s r_H} (r_s - r_H) - 2\pi\sigma r_s^2 \sqrt{N(r_s)} \right) \right) \right]\end{aligned}\tag{84}$$

for $g \geq 2$. For $g = 0$ the $m = q = 0$ adS space has no event horizon. The matching condition implies that $2\pi\sigma l = 1$ with an arbitrary matching radius r_c . The reference creation rate then depends upon this additional arbitrary parameter, and it is unclear that a meaningful comparison can be made.

Another alternative would be to compare the creation rate of black holes of genus g to the creation rate of a pure domain wall with completely empty genus $g = 0$ adS space on either side. This action I_B for this latter situation is given by [17]

$$I_B = \frac{\pi l^2}{3\sqrt{3}} \left(\frac{\sigma l}{2} \sinh^3(\sqrt{3}\frac{r_c}{l}) - \frac{1}{36} \sinh(3\sqrt{3}\frac{r_c}{l}) + \frac{r_c}{4\sqrt{3}l} \cosh(3\sqrt{3}\frac{r_c}{l}) - \frac{3}{4} e^{-\sqrt{3}r_c/l} \right)\tag{85}$$

where $r_c = l/\sqrt{(2\pi\sigma l)^2 - 1}$. The pair creation rate is then given by

$$\Gamma_{adS} = e^{-I(m,q,g)+I_B} \quad (86)$$

for a black hole of genus g .

If the cosmological constant is created from the squared field strength of a 3-form, the creation rate is relative to that for creation of a domain wall with no black holes or relativistic 3-form. In this case the relative rate is given by

$$\Gamma_{3-form} = \exp \left[(|g-1| + \delta_{g,1}) \left(2\pi\sigma r_s^2 \sqrt{V(r_s)} \beta_H - \frac{q^2}{r_s r_+} (r_s - r_H) \beta_H + \frac{s(r_s^3 - r_H^3) \beta_H}{l^2} \right) - \frac{1}{8\pi\sigma^2} \right] \quad (87)$$

for the pair creation of black holes of arbitrary genus.

The above expressions all include the case $s = 1$, which correspond to the production of static black holes in the Reissner-Nordstrom de Sitter case. If the σ -dependent terms are omitted, and r_s is taken to be the location of the cosmological horizon, then the results of ref. [2] are recovered.

6 Production of Constant Curvature Black Holes

For $b < 0$, the class of metrics (51) may be written as

$$ds^2 = (\rho^2 - \rho_+^2) \left(-(-\frac{\rho_+^2}{l^2} R^2 - k) dT^2 + \frac{dR^2}{-\frac{\rho_+^2}{l^2} R^2 - k} \right) + \frac{d\rho^2}{\frac{\rho^2 - \rho_+^2}{l^2}} + \rho^2 d\phi^2 \quad (88)$$

where without loss of generality I have set $b = -\frac{\rho_+^2}{l^2}$. The (R, T) sector is a $(1+1)$ dimensional de Sitter spacetime, with either $\partial/\partial R$ or $\partial/\partial T$ being timelike, depending on the sign of k and the magnitude of R . Choosing (again, without loss of generality) $k = -1$ yields

$$ds^2 = l^2 \left(\frac{\rho^2 - \rho_+^2}{\rho_+^2} \right) \left(-\sin^2(\theta) dt^2 + d\theta^2 \right) + \frac{d\rho^2}{\frac{\rho^2 - \rho_+^2}{l^2}} + \rho^2 d\phi^2 \quad (89)$$

as an alternate form for (88) once the coordinate transformations $R = \frac{l}{\rho_+} \cos(\theta)$ and $T = \frac{l}{\rho_+} t$ have been carried out.

Banados has recently pointed out [16] that these metrics (or alternatively the metrics (89)) can be understood as $(3 + 1)$ dimensional versions of the $(2 + 1)$ dimensional BTZ black hole; in other words, these spacetimes are $(3 + 1)$ dimensional anti de Sitter space with identifications differing from those discussed in section 3, but with the property that the identifications produced a chronological singularity that is hidden behind an event horizon. They are therefore black holes of constant curvature.

This can be understood in the following way. Consider the standard formulation of anti de Sitter spacetime, which is that of a hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = -l^2 \quad (90)$$

in a flat $(3 + 2)$ dimensional spacetime described by the metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \quad (91)$$

which has 4 rotation and 6 boost Killing vectors. Points along the orbit of the boost Killing vector $\xi = \frac{\rho_+}{l}(x_i \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_i})$ are identified under a discrete subgroup of the de Sitter group, where x_i is any one of the spacelike coordinates. Points in the region where $\xi^2 \leq 0$ contain closed timelike curves; hence $\xi^2 = 0$ is a chronological singularity. This singularity is a hyperboloid

$$x_0^2 - x_1^2 - x_2^2 = l^2 \quad (92)$$

if $x_i = x_3$ has been chosen for the boost direction. The surface for which

$$x_0^2 - x_1^2 - x_2^2 = 0 \quad (93)$$

may be regarded as a horizon: within this region timelike geodesics inevitably encounter the chronological singularity, whereas outside it they do not. The topology of this surface is a null conoid instead of a null line.

Making the coordinate transformation

$$\begin{aligned} x_\alpha &= \frac{2ly_\alpha}{1-y^2} \\ x_3 &= l \frac{1+y^2}{1-y^2} \sinh\left(\frac{\rho_+}{l}\phi\right) \\ x_4 &= l \frac{1+y^2}{1-y^2} \cosh\left(\frac{\rho_+}{l}\phi\right) \end{aligned} \quad (94)$$

with $\alpha = \{0, 1, 2\}$ transforms the metric (91) to

$$ds^2 = \frac{4l^2 dy \cdot dy}{1 - y^2} + \rho_+^2 \frac{1 + y^2}{1 - y^2} d\phi^2 \quad (95)$$

where $dy \cdot dy \equiv dy_1^2 + dy_2^2 - dy_0^2$ and $y^2 \equiv y_1^2 + y_2^2 - y_0^2$. The ranges of the coordinates are $-\infty < y_\alpha < \infty$ and $-\infty < \phi < \infty$, where $|y^2| < 1$. The black hole spacetime results upon identifying $\phi \sim \phi + 2\pi n$. The above metric can be viewed as the Kruskal form of the black hole (89), with the singularity at $y^2 = -1$, the horizon at $y^2 = 0$ and (timelike) infinity at $y^2 = 1$. By setting $\rho = \rho_+^2 \frac{1+y^2}{1-y^2}$, and choosing coordinates so that

$$y_0 = f(\rho) \sin \theta \sinh t \quad y_1 = f(\rho) \sin \theta \cosh t \quad y_2 = f(\rho) \cos \theta \quad (96)$$

where $f(\rho) = \sqrt{(\rho - \rho_+)/(\rho + \rho_+)}$, the metric (95) may be shown to be equivalent to the metric (89).

The event horizon is therefore the direct product of a circle with a null conoid, in contrast to the case I metrics, for which the horizon is the product of a genus- g 2-surface with a null line. Analytically continuing $t \rightarrow i\tau$ in (89), the null conoid becomes a 2-sphere, and the coordinate τ must be periodic with period $\beta_\rho = 2\pi l/\rho_+$ to remove conical singularities in the (τ, ρ, θ) section. The instanton again flares out like a solid bell, but with topology $R^3 \times S^1$. Matching two copies of this manifold together at a radius r at the open end of their bells yields a Riemannian section with topology $S^3 \times S^1$. There is a single domain wall of topology $S^2 \times S^1$ and two bolts of topology S^1 where the Killing field $\frac{\partial}{\partial \tau}$ vanishes. The nucleation surface Σ which joins the Lorentzian and Riemannian sections is located along the $\tau = 0$ and $\tau = \beta_\rho/2$ segments.

The matching condition differs from that given for the case I metrics. The Lanczos conditions yield

$$\left[h'(\rho) + \frac{h(\rho)}{\rho} \right] \Big|_{\rho=\rho_s} = 4\pi\sigma \quad (97)$$

where $h^2(\rho) = \frac{\rho^2 - \rho_+^2}{l^2}$ and only static solutions at $\rho = \rho_s$ are being considered. Solving (97) for ρ_s yields

$$\rho_s^2 = \frac{\rho_+^2}{2} \left[1 + \frac{2\pi\sigma l}{\sqrt{(2\pi\sigma l)^2 - 1}} \right] \quad (98)$$

where ρ_s must be positive. Matching is only possible provided $l\sigma$ is sufficiently large, $2\pi l\sigma > 1$.

These instantons are uncharged, and so the action is easily calculated to be

$$\begin{aligned} I_{cc} &= \int d^4x \sqrt{g} \frac{3}{8\pi l^2} - \frac{\sigma}{2} \int_W \sqrt{h} d^3x \\ &= \frac{l\beta_\rho}{2\rho_+^2} \sqrt{(\rho_s^2 - \rho_+^2)^3} - \rho_s \frac{\pi\sigma l^2 \beta_\rho}{\rho_+^2} (\rho_s^2 - \rho_+^2) \end{aligned} \quad (99)$$

with ρ_s given by (98). The production rate of these constant curvature black holes is then given by

$$\Gamma_{adS} = e^{-I_{cc} + I_B} \quad (100)$$

relative to the creation rate of a pure domain wall with completely empty adS space on either side, with I_B given by (85).

7 Conclusions

By requiring the cosmological c-metric to be free of conical singularities in the (x, ϕ) section, a large variety of candidate black hole spacetimes emerged. There were three classes of such spacetimes. The first (case I) included black holes whose event horizons are compact 2-surfaces with topology of arbitrary genus g . For $g = 0$, these black holes are the usual Reissner-Nordstrom (anti) de Sitter type. For $g \geq 1$, they are all asymptotically Reissner-Nordstrom anti de Sitter ($\Lambda < 0$), with the entire spacetime inheriting the topology of the event horizon. For genus $g \geq 2$ solutions with both positive and negative mass are permitted. The second class (case II) all have $\Lambda < 0$, and the event horizons are non-compact 2-surfaces. The third class (case III) of metrics are all asymptotic to the constant curvature black holes (88), but contain naked singularities unless $m = q = 0$. For $m = q = 0$ these metrics are the constant curvature black holes discussed by Banados [16].

Black hole pairs of both the case I and case III classes may be produced using domain walls, although this is not necessary if $\Lambda > 0$ [2]. For case I metrics, if $\Lambda < 0$, then the only allowed solutions are static unless the topology of the event horizon is spherical ($g = 0$). The production rates of these topological black holes calculated in section 5 are for black holes of a given genus g . In general, the larger the genus, the more suppressed the

production rate. More generally one could compute the rate for producing black hole pairs of all possible topologies. This involves a simple sum over the genus which yields

$$\Gamma = e^{-I(m,q,0)} + e^{-I(m,q,1)} + \frac{e^{-2\hat{I}(m,q)}}{1 - e^{-\hat{I}(m,q)}} \quad (101)$$

for the (unnormalized) inclusive production rate, where

$$\hat{I}(m, q) = 2\pi\sigma r_s^2 \sqrt{V(r_s)}\beta_H - \frac{q^2}{r_s r_+}(r_s - r_H)\beta_H - \frac{(r_s^3 - r_H^3)\beta_H}{l^2} \quad (102)$$

In $N = 1$ supergravity theories, domain walls naturally arise as boundaries between regions of isolated vacua of the supergravity matter fields. There is no *a-priori* reason to exclude walls of a given topology. A wall of a specified topology will in general be quantum mechanically unstable to pair creation of black holes of the same topology, as the preceding arguments in this paper demonstrate. The extension of these arguments to situations in which rotation, dilatonic couplings, and charged domain walls are included remain interesting open questions.

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